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Hyperbolic heat conduction and local equilibrium: a second law analysis

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Abstract—An analysis of the compatibility of Cattaneo–Vernotte's constitutive equation for the heat flux density vector with the hypothesis of local thermodynamic equilibrium is presented. This compatibility is checked by determining the entropy production rate per unit volume. In fact, within the scheme of local equilibrium, Clausius' inequality implies that the entropy production rate must be non-negative. The evaluation of the entropy production rate is performed for a solid slab which experiences a sudden change of the boundary temperature and for a semi-infinite solid with a time-varying boundary heat flux. A domain of non-stationary phenomena is found in which Cattaneo–Vernotte's equation is compatible with the assumption of local equilibrium and yields results definitely different from those predicted by Fourier's equation. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

Two constitutive equations are used, nowadays, in the analysis of conduction heat transfer. The first, and most widely employed, is Fourier's law

$$\mathbf{q} = -k\nabla T \tag{1}$$

where $\mathbf{q}$ is the heat flux density vector, k is a positive scalar called thermal conductivity and T is the local equilibrium temperature. As it is well known, equation (1) yields a parabolic differential equation for the temperature field and, as a consequence, an infinite propagation speed of thermal signals. This feature appears as unsatisfactory. On the other hand, as it will be proved in next section, equation (1) is always perfectly compatible with the scheme of local equilibrium, because, within this scheme, it yields an expression of the local entropy production rate which can assume only positive values.

To overcome the paradox of an infinite propagation speed of thermal signals, Cattaneo and Vernotte [1, 2] have proposed a different constitutive equation for conduction heat transfer, namely

$$\mathbf{q}+\tau\frac{\partial \mathbf{q}}{\partial t} = -k\nabla T \tag{2}$$

where τ is a positive scalar called thermal relaxation time.

Equation (2) yields a hyperbolic differential equation for the temperature field and eliminates the paradox of an infinite propagation speed of thermal signals. Obviously, equation (2) reduces to equation (1) for $\tau = 0$. Some experimental evidence in favour of equation (2), with respect to equation (1), is already available, especially at very low temperatures. For instance, the propagation speed of thermal signals has been measured in liquid helium [3], in NaF at about 10 K [4], in Bi at 3.4 K [5]. Recent measurements of the propagation speed of thermal signals at room temperatures, in unhomogeneous materials, have been performed by Kaminski [6]. According to this author, some unhomogeneous materials, such as sand and $NaHCO_3$, have relaxation times of about 20 s.

These experiments seem to suggest that equation (1) is only an approximation of equation (2), valid when the product $\tau(\partial \mathbf{q}/\partial t)$ is negligible. Many solutions of heat conduction problems based on equation (2) are available in the literature. An interesting review of these solutions is provided by Özisik and Tzou [7]. However, deep problems arise when the thermodynamic interpretation of the consequences of equation (2) is considered. In fact, if equation (2) is analysed within the framework of the traditional irreversible thermodynamics, which is based on the assumption of local equilibrium, then it yields an expression of the entropy production rate which can assume both positive and negative values. Since negative values of the entropy production rate are in contrast with Clausius' inequality, as it will be formally proved in the next section, some authors have conceived an extended framework of irreversible thermodynamics, in which the assumption of local equilibrium is released. This framework, called extended irreversible thermodynamics, is built so that equation (2) yields an expression of the entropy production rate which can assume only positive values. However, a widely accepted theory of extended irreversible thermodynamics, based on rigorous definitions, internally consistent and endowed with some experimental verification, is not yet available.

Indeed, different theories of extended irreversible

NOMENCLATURE

a	dimensionless function of η defined in Appendix A
a_n	dimensionless functions of η defined in Appendix A
A	thermodynamic system
A_1, A_2	thermodynamic states of A
b_n	dimensionless coefficients defined in Appendix A
c_n	dimensionless coefficients defined in Appendix A
c_v	specific heat at constant volume
F	dimensionless function of time defined by equation (37)
g	function of x and t defined in Appendix B
I_n	modified Bessel function of first kind and order n
k	thermal conductivity
L	one half of the slab thickness
$\mathscr{L}^{-1}$	inverse Laplace transform operator
$\mathbf{n}$	outward unit normal
n	natural number
p	Laplace transformed variable
q	x-component of the heat flux density
Q	constant power per unit area
$\mathbf{q}$	heat flux density
s	entropy per unit mass
s_0	entropy per unit mass at equilibrium
S	entropy
t	time
t_0, t_1, t_2, t_3	instants of time
t_p	time constant which determines the time scale of the pulse function $F(t)$
T	temperature
T_0	initial temperature
T_w	boundary temperature
u	energy per unit mass
u_0	energy per unit mass at equilibrium
u_a	fixed value of the energy per unit mass
U	Heaviside's unit step function
V	region of space
∂V	boundary surface of V
x	spatial coordinate
$\mathbf{x}$	position vector.

Greek symbols

α	$=k/(\rho c_v)$, thermal diffusivity
β	dimensionless constant employed in Appendix A
β_n	eigenvalues defined in Appendix A
γ_n	dimensionless constants defined in Appendix A
δ_n	dimensionless constants defined in Appendix A
δ	Dirac's delta distribution
δQ	infinitesimal heat quantity employed in equation (10)
η	dimensionless time defined by equation (25)
ϑ	dimensionless temperature defined by equation (25)
Θ	nonequilibrium temperature defined by equation (7)
κ	function defined by equation (5)
Λ	dimensionless parameter defined by equation (25)
ξ	dimensionless coordinate defined by equation (25)
ρ	mass density
σ	entropy production rate per unit volume
τ	thermal relaxation time
φ	dimensionless entropy production rate per unit volume defined by equation (25)
ϕ	dimensionless function of $\bar{t}$ defined by equation (38)
$\dot{\phi}$	derivative of ϕ with respect to $\bar{t}$
χ	dimensionless heat flux density in the x-direction defined by equation (25)
ψ	Helmholtz free energy per unit mass
ω	Lagrange multiplier employed in equation (5).

Superscripts

$\tilde{}$	Laplace transformed functions
$\bar{}$	dimensionless quantities defined by equation (38)
$'$	dummy integration variables.

thermodynamics have been presented. A review of these theories can be found in refs. [8, 9]. Some theories of extended irreversible thermodynamics will be discussed here.

Gurtin and Pipkin [10] propose a theory of extended irreversible thermodynamics which is based on the assumption that the specific Helmholtz free energy ψ, the specific entropy s and the heat flux vector $\mathbf{q}$ depend on temperature T, on the history of T and on the history of ∇T. No definition of the specific nonequilibrium entropy s and of the nonequilibrium temperature T is given. Moreover, no explicit form of the dependence of ψ, s and $\mathbf{q}$ on T and on the histories of T and of ∇T is proposed. Therefore, this treatment of extended irreversible thermodynamics does not appear as an experimentally verifiable physical theory.

Coleman *et al.* [11] present a treatment of extended irreversible thermodynamics the aim of which is to prove that equation (2) is compatible with the second law of thermodynamics only if the specific internal energy u does not depend only on T, but also on $\mathbf{q}$. More precisely, the authors obtain expressions of u and s for a material at constant mass density. If the material is isotropic and has constant values of the thermal relaxation time and of the thermal conductivity, these expressions assume the form

$$u(T,\mathbf{q}) = u_0(T) + \frac{\tau}{\rho k T}\mathbf{q}\cdot\mathbf{q} \tag{3}$$

$$s(T,\mathbf{q}) = s_0(T) + \frac{\tau}{2\rho k T^2}\mathbf{q}\cdot\mathbf{q} \tag{4}$$

where $u_0(T)$ is the specific internal energy at equilibrium and $s_0(T)$ is the specific entropy at equilibrium. It is easily proved that equations (3) and (4) are incompatible with the highest-entropy principle. For a closed simple system at constant volume, this principle can be stated as follows [12]: among the states of the system with a fixed value of energy, the entropy takes on its maximum value in the unique stable equilibrium state which corresponds to the given energy value. In order to check if equations (3) and (4) are compatible with the highest-entropy principle, one must verify that the function

$$\kappa(T,\mathbf{q},\omega) = s(T,\mathbf{q}) + \omega[u(T,\mathbf{q}) - u_a] \tag{5}$$

where ω is a Lagrange multiplier and u_a is a fixed value of the specific internal energy, has a maximum at equilibrium, i.e. for $\mathbf{q} = 0$. The partial derivatives of $\kappa(T,\mathbf{q},\omega)$ are zero at equilibrium if $\omega = -1/T$. However, if the equilibrium value of the specific heat at constant volume is independent of temperature, the Hessian matrix of $\kappa(T,\mathbf{q},\omega)$ has one positive eigenvalue and four negative eigenvalues for $\mathbf{q} = 0$ and $\omega = -1/T$. Therefore, the function $\kappa(T,\mathbf{q},\omega)$ has neither a maximum nor a minimum at equilibrium, in contrast with the highest-entropy principle. As a consequence, equations (3) and (4) are incompatible with the laws of thermodynamics. Moreover, the thermodynamic temperature T, in local nonequilibrium states, appears as undefined.

A more consistent theory of extended irreversible thermodynamics is presented by Jou *et al.* [13]. The authors assume that, for the local nonequilibrium states of a medium with uniform and constant density, the specific entropy is a function of the specific internal energy and of the heat flux density vector, i.e.

$$s = s(u,\mathbf{q}). \tag{6}$$

Then, they define the nonequilibrium temperature Θ as

$$\frac{1}{\Theta} = \left(\frac{\partial s}{\partial u}\right)_{\mathbf{q}} \tag{7}$$

and postulate that the entropy flux vector is given by $\mathbf{q}/\Theta$ and that the local entropy production is positive. They prove that Cattaneo-Vernotte's constitutive equation, expressed in the form

$$\mathbf{q} + \tau\frac{\partial \mathbf{q}}{\partial t} = -k\nabla\Theta \tag{8}$$

is compatible with the assumptions stated above. Moreover, they obtain the following explicit expression of equation (6):

$$s(u,\mathbf{q}) = s_0(u) - \frac{\tau}{2\rho k T^2}\mathbf{q}\cdot\mathbf{q} \tag{9}$$

where $s_0(u)$ is the specific entropy at equilibrium, for the given value of u, and T is the equilibrium thermodynamic temperature, defined as $T^{-1} = \mathrm{d}s_0/\mathrm{d}u$. The compatibility of equation (8) with the assumptions made by Jou *et al.* [13] has been proved also by Tzou [14].

The theory of extended irreversible thermodynamics proposed by Jou *et al.* [13] has no internal inconsistency. However, the theory is based on assumptions which cannot undergo a direct experimental verification. In fact, no definition of the nonequilibrium specific entropy is available in thermodynamics. The entropy difference between two states A_1 and A_2 of any system A is defined as [15]

$$S(A_2) - S(A_1) = \int_{A_1}^{A_2}\frac{\delta Q}{T} \tag{10}$$

where δQ is the infinitesimal heat quantity that the system receives from a reservoir with temperature T and the integral must be evaluated along any reversible process of system A from A_1 to A_2. As is stated by Fermi [15], a reversible process of system A from A_1 to A_2 exists only if A_1 and A_2 are either stable equilibrium states or local equilibrium states. In the second case, system A can be considered as a collection of infinite subsystems in stable equilibrium which can be brought reversibly, one by one, from their initial to their final state. Since the specific entropy of a system which is not in local stable equilibrium cannot be defined by means of equation (10), equation (9) cannot be considered as a verifiable result.

A partial experimental verification of the theory proposed by Jou *et al.* [13] could be based on measurements of the difference between Θ and T. Although an experiment with this objective has been conceived by Jou and Casas-Vázquez [16], no experimental evidence supporting the theory proposed by Jou *et al.* [13] is still available.

To summarize, some theories of extended irreversible thermodynamics which disagree with each other have been presented. Some of these theories are internally inconsistent, others cannot be verified, others again can be partially verified, in principle, but still lack any experimental support. Therefore, it appears interesting to check if a domain exists, in which the constitutive equation (2) proposed by Cattaneo and Vernotte can be interpreted within the usual framework of irreversible thermodynamics, which is based on the assumption of local equilibrium. The aim of this paper is to develop this analysis. It will be proved that a domain exists, in heat conduction phenomena, in which equation (2) yields results that differ from those obtainable with equation (1) and a physical interpretation of the results within the scheme of local equilibrium is still possible. It will also be proved that, out of this domain, equation (2) becomes incompatible with the scheme of local equilibrium.

MATHEMATICAL MODEL

In this section, the mathematical description of hyperbolic heat conduction in solid media is outlined. Moreover, an expression of the entropy production rate σ based on the local equilibrium hypothesis is determined and a proof that σ must be non-negative is presented.

If no heat generation is present within a solid and the mass density ρ is constant, the local energy balance equation can be expressed as

$$\nabla\cdot\mathbf{q}+\rho\frac{\partial u}{\partial t}=0. \tag{11}$$

If the local equilibrium hypothesis holds, any sufficiently small volume element of the solid can be considered in stable equilibrium at any instant of time. Moreover, the specific internal energy u depends only on T, so that $\mathrm{d}u = c_{\mathrm{v}}\,\mathrm{d}T$. Therefore, if the thermal conductivity k, the thermal diffusivity α, the specific heat at constant volume c_{v} and the thermal relaxation time τ of the material are considered as constants, equations (2) and (11) yield both

$$\alpha\nabla^2 T=\frac{\partial T}{\partial t}+\tau\frac{\partial^2 T}{\partial t^2} \tag{12}$$

and

$$\alpha\nabla(\nabla\cdot\mathbf{q})=\frac{\partial \mathbf{q}}{\partial t}+\tau\frac{\partial^2 \mathbf{q}}{\partial t^2}. \tag{13}$$

Equation (12) allows a direct determination of the temperature field, while the heat flux density vector can be directly determined by equation (13). If the temperature field is known, the heat flux density vector can be determined also by integrating equation (2) with respect to time, namely

$$\mathbf{q}(\mathbf{x},t)=\mathrm{e}^{-(t-t_0)/\tau}\,\mathbf{q}(\mathbf{x},t_0)-\frac{k}{\tau}\int_{t_0}^{t}\mathrm{e}^{\frac{t'-t}{\tau}}\,\nabla T(\mathbf{x},t')\,\mathrm{d}t'. \tag{14}$$

The local entropy balance equation can be written as

$$\nabla\cdot\left(\frac{\mathbf{q}}{T}\right)+\rho\frac{\partial s}{\partial t}=\sigma. \tag{15}$$

Since ρ = constant and the relation $\mathrm{d}u = T\,\mathrm{d}s$ holds locally, equations (11) and (15) yield the expression of the entropy production rate

$$\sigma=-\frac{1}{T^2}\mathbf{q}\cdot\nabla T. \tag{16}$$

On account of Clausius' inequality, the entropy production rate must be non-negative, as it will be proved in the following. Let us consider a portion of solid A contained in the volume V with boundary ∂V, such that in the time interval $[t_1, t_2]$ system A changes its thermodynamic state from A_1 to A_2. If $\mathbf{n}$ is the outward unit normal of ∂V, equation (15) yields

$$\int_{t_1}^{t_2}\int_{\partial V}\frac{\mathbf{q}\cdot\mathbf{n}}{T}\,\mathrm{d}^2x\,\mathrm{d}t+S(A_2)-S(A_1)=\int_{t_1}^{t_2}\int_{V}\sigma\,\mathrm{d}^3x\,\mathrm{d}t. \tag{17}$$

On the other hand, since A_1 and A_2 are local equilibrium states, it is possible to perform a reversible time evolution in the interval $[t_2, t_3]$ which brings back system A to its initial state, A_1 [15]. For this second evolution, equation (15) yields

$$\left(\int_{t_2}^{t_3}\int_{\partial V}\frac{\mathbf{q}\cdot\mathbf{n}}{T}\,\mathrm{d}^2x\,\mathrm{d}t\right)_{\text{reversible}}+S(A_1)-S(A_2)=0. \tag{18}$$

On account of equations (17) and (18), one obtains

$$\int_{t_1}^{t_2}\int_{\partial V}\frac{\mathbf{q}\cdot\mathbf{n}}{T}\,\mathrm{d}^2x\,\mathrm{d}t+\left(\int_{t_2}^{t_3}\int_{\partial V}\frac{\mathbf{q}\cdot\mathbf{n}}{T}\,\mathrm{d}^2x\,\mathrm{d}t\right)_{\text{reversible}}=\int_{t_1}^{t_2}\int_{V}\sigma\,\mathrm{d}^3x\,\mathrm{d}t. \tag{19}$$

The left hand side of equation (19) can be expressed as a time integral over a cyclic time evolution of system A, which starts and ends at state A_1. Therefore, equation (19) can be rewritten as

$$\oint_{A_1\to A_1}\int_{\partial V}\frac{\mathbf{q}\cdot\mathbf{n}}{T}\,\mathrm{d}^2x\,\mathrm{d}t=\int_{t_1}^{t_2}\int_{V}\sigma\,\mathrm{d}^3x\,\mathrm{d}t. \tag{20}$$

Indeed, $\mathbf{q}\cdot\mathbf{n}\,\mathrm{d}^2x\,\mathrm{d}t$ represents the opposite of the infinitesimal quantity of heat which enters system A through the surface element d^2x in a time interval with duration $\mathrm{d}t$. Therefore, Clausius' inequality implies that the left hand side of equation (20) is non-negative, so that the right hand side of equation (20) is also non-negative. Since both the portion of solid A and the time interval $[t_1, t_2]$ have been chosen arbitrarily, the entropy production rate σ is non-negative at every point and at every instant of time.

It has been proved that the hypothesis of local equilibrium implies that σ is non-negative at every point and at every instant of time. Therefore, if a state of the solid medium is such that the entropy production fails to be non-negative everywhere, one can conclude that the state under exam is not a local equilibrium state.

On account of equations (2) and (16), the entropy production rate can be expressed as

$$\sigma = \frac{1}{kT^2}\left(\mathbf{q}\cdot\mathbf{q}+\tau\mathbf{q}\cdot\frac{\partial\mathbf{q}}{\partial t}\right). \tag{21}$$

Whenever the heat flux density $\mathbf{q}$ has a constant direction and decreases at some point so steeply that $|\partial\mathbf{q}/\partial t| > |\mathbf{q}|/\tau$, the right hand side of equation (21) becomes negative and the local equilibrium scheme cannot be applied. If this circumstance occurs, the temperature field which can be evaluated by solving equation (12) cannot be endowed with its usual physical meaning and equation (12) is incompatible with local equilibrium thermodynamics.

ON TAITEL'S PARADOX

In this section, hyperbolic heat conduction in a parallel and infinitely wide slab which undergoes a sudden temperature change at its boundary is studied. The temperature field and the entropy production rate are determined.

Let us consider heat conduction in an infinitely wide parallel slab with thickness $2L$ such that the thermal conductivity k, the thermal diffusivity α, the specific heat at constant volume c_v and the thermal relaxation time τ of the slab can be considered as constant. Throughout the slab, at time $t = 0$, the heat flux density $\mathbf{q}$ is zero and the temperature field T is uniform with value T_0. As a consequence, also $\partial T/\partial t$ is zero at time $t = 0$, as can be easily proved by employing equation (11). For every $t > 0$, the temperature distribution on the two sides of the slab is kept uniform with a value $T_w \neq T_0$.

This problem has been analysed by Taitel [17] in the framework of hyperbolic heat conduction. Taitel obtains the transient temperature field and points out that the absolute value of the temperature change $T-T_0$ may exceed $|T_w-T_0|$. A thermodynamic interpretation of this paradoxical feature can be based on the evaluation of the entropy production rate. Since Taitel's method to determine the temperature field is rather involved and the mathematical expression of the temperature distribution is very complicated, a new solution of equation (12) can be obtained for this problem by a simple separation of variables.

By considering an axial coordinate x orthogonal to the slab and such that the two sides of the slab are at positions $x = -L$ and $x = L$ respectively, the governing equations can be expressed as

$$\alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}+\tau\frac{\partial^2 T}{\partial t^2} \tag{22}$$

$$T(x,0) = T_0, \quad \left.\frac{\partial T}{\partial t}\right|_{t=0} = 0 \tag{23}$$

$$T(-L,t>0) = T(L,t>0) = T_w. \tag{24}$$

By introducing the dimensionless quantities

$$\xi = \frac{x}{2L} \quad \eta = \frac{\alpha t}{4L^2} \quad \vartheta = \frac{T_w-T}{T_w-T_0} \quad \chi = \frac{2Lq}{k(T_w-T_0)}$$

$$\varphi = \frac{4L^2T^2}{k(T_w-T_0)^2}\sigma \quad \Lambda = \frac{\alpha\tau}{4L^2} \tag{25}$$

equations (22)–(24) can be rewritten as

$$\frac{\partial^2\vartheta}{\partial\xi^2} = \frac{\partial\vartheta}{\partial\eta}+\Lambda\frac{\partial^2\vartheta}{\partial\eta^2} \tag{26}$$

$$\vartheta(\xi,0) = 1 \quad \left.\frac{\partial\vartheta}{\partial\eta}\right|_{\eta=0} = 0 \tag{27}$$

$$\vartheta(-1/2,\eta>0) = \vartheta(1/2,\eta>0) = 0. \tag{28}$$

In Appendix A, equations (26)–(28) are solved by a separation of variables and the dimensionless temperature is expressed as

$$\vartheta(\xi,\eta) = \sum_{n=0}^{\infty}(b_n e^{\gamma_n\eta}+c_n e^{\delta_n\eta})\cos(\beta_n\xi). \tag{29}$$

The eigenvalues β_n and the coefficients b_n, c_n, γ_n, δ_n are also evaluated in Appendix A. On account of equation (14) and of the initial condition $q(x,0) = 0$, one obtains

$$q(x,t) = -\frac{k}{\tau}\int_0^t e^{(t'-t)/\tau}\frac{\partial T(x,t')}{\partial x}\,dt'. \tag{30}$$

Equations (25) and (30) yield

$$\chi(\xi,\eta) = \frac{1}{\Lambda}\int_0^\eta e^{(\eta'-\eta)/\Lambda}\frac{\partial\vartheta(\xi,\eta')}{\partial\xi}\,d\eta'. \tag{31}$$

By employing equations (29) and (31), one obtains

$$\chi(\xi,\eta) = -\frac{e^{-\eta/\Lambda}}{\Lambda}\sum_{n=0}^{\infty}\beta_n\left\{\frac{b_n}{\gamma_n+1/\Lambda}[e^{(\gamma_n+1/\Lambda)\eta}-1]+\frac{c_n}{\delta_n+1/\Lambda}[e^{(\delta_n+1/\Lambda)\eta}-1]\right\}\sin(\beta_n\xi). \tag{32}$$

Moreover, equations (16) and (25) yield

$$\varphi(\xi,\eta) = \chi(\xi,\eta)\frac{\partial\vartheta(\xi,\eta)}{\partial\xi}. \tag{33}$$

The dimensionless entropy production rate can be evaluated by equations (29), (32) and (33). Equation (33) ensures that, due to the symmetry of the problem, the dimensionless entropy production rate will always be zero at $\xi = 0$.

Plots of $(1-\vartheta)$ and φ vs ξ are reported in Fig. 1, for $\Lambda = 1$ and $\eta = 0.7$. These plots reveal that $|T-T_0|$ may exceed $|T_w-T_0|$ as pointed out by Taitel [17]. However, in the case examined, a negative entropy production rate is present at every point within the slab. As a consequence, the local equilibrium scheme cannot be employed in the case represented in Fig. 1. Therefore, T does not represent the thermodynamic temperature and no violation of the laws of ther-

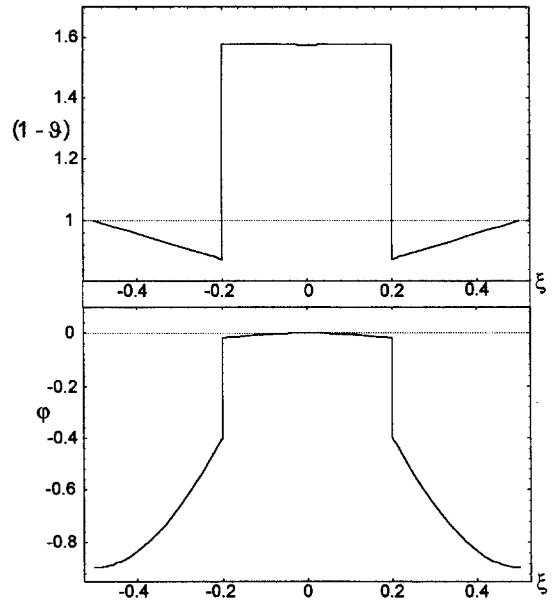

Fig. 1. Plots of $1-\vartheta=(T-T_0)/(T_w-T_0)$ vs ξ and of φ vs ξ, for $\Lambda=1$ and $\eta=0.7$, obtained by employing equations (29), (32) and (33).

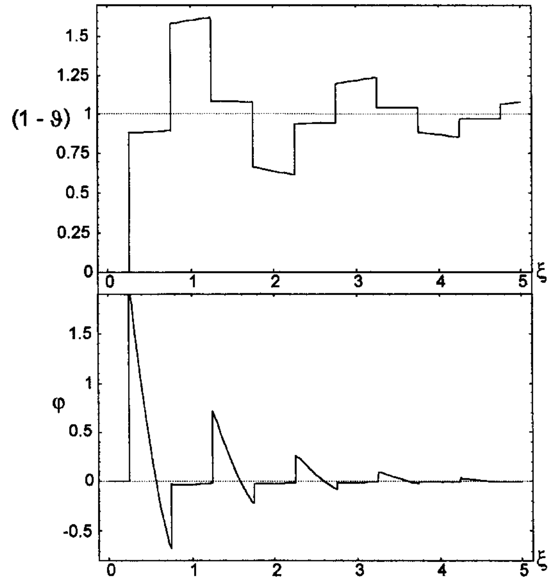

Fig. 2. Plots of $1-\vartheta=(T-T_0)/(T_w-T_0)$ vs η and of φ vs η, for $\Lambda=1$ and $\xi=0.25$, obtained by employing equations (29), (32) and (33).

modynamics occurs. Incidentally, let us note that at the positions where the temperature field is discontinuous the entropy production rate presents Dirac-delta singularities, as can be inferred from equation (33). These singularities have not been represented in Fig. 1.

In Fig. 2, plots of $(1-\vartheta)$ and φ vs η are reported for $\Lambda=1$ and $\xi=0.25$. These plots show that the equilibrium value for $(1-\vartheta)$, i.e. one, is attained by an oscillatory process during which $(1-\vartheta)$ is alternately greater and smaller than one and the entropy production rate is alternately positive and negative. This behaviour is quite different from that predicted in parabolic heat conduction and described, for instance, in Carslaw and Jaeger [18]. In fact, parabolic heat conduction predicts that $(1-\vartheta)$ continuously increases from zero to one at any internal position ξ.

A SEMI-INFINITE SOLID WITH A PRESCRIBED BOUNDARY HEAT FLUX

In this section, hyperbolic heat conduction in a semi-infinite solid with a time-varying boundary heat flux is considered. The heat flux density, the entropy production rate per unit volume and the temperature field are determined.

Let us consider heat conduction in a semi-infinite solid bounded by a plane surface and such that the thermal conductivity k, the thermal diffusivity α, the specific heat at constant volume c_v and the thermal relaxation time τ of the material can be considered as constant. Let us denote by x the spatial coordinate along the axis orthogonal to the boundary surface of the solid, directed inside the solid and such that $x=0$ represents the position of the boundary surface. Throughout the solid at time $t=0$, the heat flux density $\mathbf{q}$ is zero and the temperature field T is uniform with value T_0. As a consequence, also $\partial\mathbf{q}/\partial t$ is zero at time $t=0$, as can be easily proved by employing equation (2). For $t>0$, the x-component of the heat flux density vector at the boundary surface of the solid is given by $q(0,t)=QF(t)$, where F is a differentiable function of t such that $F(0)=0$. On account of equation (21), the entropy production rate at $x=0$ is given by

$$\sigma(0,t)=\frac{Q^2}{kT(0,t)^2}F(t)\left[F(t)+\tau\frac{\mathrm{d}F(t)}{\mathrm{d}t}\right]. \qquad (34)$$

Let us suppose that $F(t)$ behaves as a pulse which initially increases, reaches a maximum and then decreases without undergoing sign changes. Equation (34) ensures that $\sigma(0,t)$ can become negative if $F(t)$ decreases faster than $e^{-t/\tau}$.

The heat flux density component along the direction x can be determined by solving equation (13) with its boundary and initial conditions

$$\alpha\frac{\partial^2 q}{\partial x^2}=\frac{\partial q}{\partial t}+\tau\frac{\partial^2 q}{\partial t^2} \qquad (35)$$

$$q(x,0)=0 \quad \left.\frac{\partial q}{\partial t}\right|_{t=0}=0 \qquad (36)$$

$$q(0,t>0)=QF(t). \qquad (37)$$

Equations (35)–(37) are solved in Appendix B by the Laplace transform method. If one defines the dimensionless quantities

$$\bar{x} = \frac{x}{\sqrt{\alpha t_\mathrm{p}}} \quad \bar{t} = \frac{t}{t_\mathrm{p}} \quad \bar{\tau} = \frac{\tau}{t_\mathrm{p}} \quad \bar{q} = \frac{q}{Q} \quad \phi(\bar{t}) = F(t_\mathrm{p}\bar{t})$$

$$\bar{\sigma} = \frac{kT^2}{Q^2}\sigma \quad \bar{T} = k\frac{T-T_0}{Q\sqrt{\alpha t_\mathrm{p}}} \tag{38}$$

the solution of equations (35)–(37) can be written in the dimensionless form

$$\bar{q}(\bar{x},\bar{t}) = U(\bar{t}-\bar{x}\sqrt{\bar{\tau}}) \times\left[\frac{\bar{x}}{2\sqrt{\bar{\tau}}}\int_{\bar{x}\sqrt{\bar{\tau}}}^{\bar{t}} \frac{\mathrm{e}^{-\bar{t}'/(2\bar{\tau})}\phi(\bar{t}-\bar{t}')}{\sqrt{\bar{t}'^2-\bar{x}^2\bar{\tau}}} I_1\left(\frac{1}{2\bar{\tau}}\sqrt{\bar{t}'^2-\bar{x}^2\bar{\tau}}\right)\mathrm{d}\bar{t}' + \phi(\bar{t}-\bar{x}\sqrt{\bar{\tau}})\mathrm{e}^{-\bar{x}/(2\sqrt{\bar{\tau}})}\right]. \tag{39}$$

Moreover, on account of equations (21) and (38), the dimensionless entropy production rate can be written as

$$\bar{\sigma}(\bar{x},\bar{t}) = \bar{q}(\bar{x},\bar{t})\left[\bar{q}(\bar{x},\bar{t})+\bar{\tau}\frac{\partial\bar{q}(\bar{x},\bar{t})}{\partial\bar{t}}\right]. \tag{40}$$

On account of equations (39) and (40), both $\bar{q}(\bar{x},\bar{t})$ and $\bar{\sigma}(\bar{x},\bar{t})$ become non-zero only when $\bar{t} > \bar{x}\sqrt{\bar{\tau}}$, i.e. when $t > x\sqrt{\tau/\alpha}$. This result is not surprising since $\sqrt{\alpha/\tau}$ represents the propagation speed of thermal waves.

At any instant of time, the limit of $T(x, t)$ for $x \to +\infty$ must be equal to T_0. Therefore, equation (2) implies that

$$T(x,t) = T_0 + \frac{1}{k}\int_x^{+\infty}\left[q(x',t)+\tau\frac{\partial q(x',t)}{\partial t}\right]\mathrm{d}x'. \tag{41}$$

On account of equations (38), (39) and (41), one obtains

$$\bar{T}(\bar{x},\bar{t}) = \int_{\bar{x}}^{+\infty}\left[\bar{q}(\bar{x}',\bar{t})+\bar{\tau}\frac{\partial\bar{q}(\bar{x}',\bar{t})}{\partial\bar{t}}\right]\mathrm{d}\bar{x}'$$

$$= \frac{1}{\sqrt{\bar{\tau}}}U(\bar{t}-\bar{x}\sqrt{\bar{\tau}})\left\{\int_{\bar{x}\sqrt{\bar{\tau}}}^{\bar{t}}\mathrm{e}^{-\bar{t}'/(2\bar{\tau})}[\phi(\bar{t}-\bar{t}') + \bar{\tau}\dot{\phi}(\bar{t}-\bar{t}')]I_0\left(\frac{1}{2\bar{\tau}}\sqrt{\bar{t}'^2-\bar{x}^2\bar{\tau}}\right)\mathrm{d}\bar{t}'\right\}. \tag{42}$$

A similar expression of the temperature field in the case of a semi-infinite solid with a periodic on-off heat flux at the boundary surface has been determined by Glass *et al.* [19] by employing a version of Duhamel's theorem extended to hyperbolic heat conduction.

Let us assume that the heat flux density at $x = 0$ is given by the following pulse function $\phi(\bar{t})$ given by

$$\phi(\bar{t}) = \tfrac{1}{4}\bar{t}^2\mathrm{e}^{-(\bar{t}-2)}. \tag{43}$$

As a consequence of equation (43), function $\phi(\bar{t})$ reaches its maximum for $\bar{t} = 2$. On account of the considerations reported at the beginning of this section, a heat pulse expressed through equation (43) is expected to give rise to negative entropy production rates if $\tau > t_\mathrm{p}$, i.e. if $\bar{\tau} > 1$. A plot of the function $\phi(\bar{t})$ given by equation (43) is reported in Fig. 3.

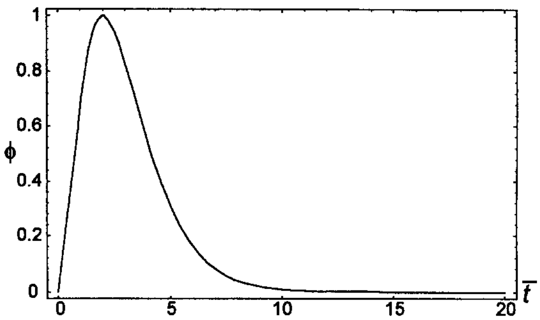

Fig. 3. Plot of ϕ vs $\bar{t}$, obtained by employing equation (43).

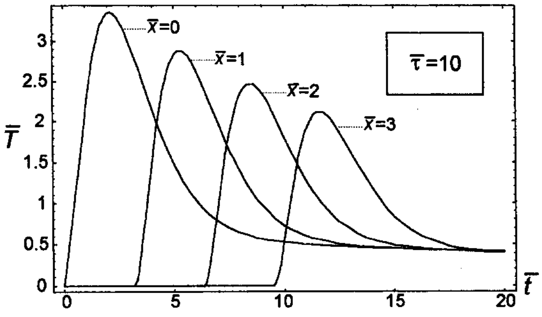

Fig. 4. Plots of $\bar{T}$ vs $\bar{t}$ at different positions and for $\bar{\tau} = 10$, obtained by employing equations (42) and (43).

By employing equations (39), (40) and (42), both the dimensionless temperature $\bar{T}$ and the dimensionless entropy production rate $\bar{\sigma}$ can be evaluated. In Fig. 4, the dimensionless temperature $\bar{T}$ vs $\bar{t}$ is reported for $\bar{\tau} = 10$ at positions $\bar{x} = 0$, $\bar{x} = 1$, $\bar{x} = 2$ and $\bar{x} = 3$. Figure 4 shows that, at different positions, the thermal signals are delayed due to the finite speed of propagation. In Fig. 5, the dimensionless entropy production rate $\bar{\sigma}$ vs $\bar{t}$ is reported for $\bar{\tau} = 10$ at positions $\bar{x} = 0$, $\bar{x} = 1$, $\bar{x} = 2$ and $\bar{x} = 3$. As expected, $\bar{\sigma}$ can be negative, so that this value of $\bar{\tau}$ is incompatible

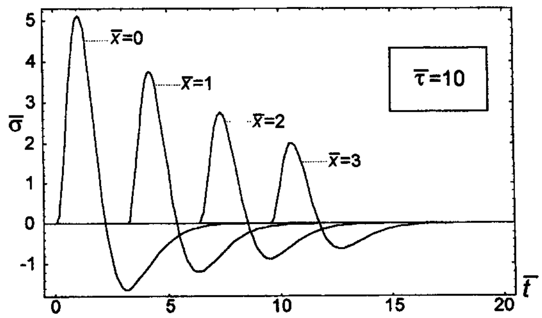

Fig. 5. Plots of $\bar{\sigma}$ vs $\bar{t}$ at different positions and for $\bar{\tau} = 10$, obtained by employing equations (39), (40) and (43).

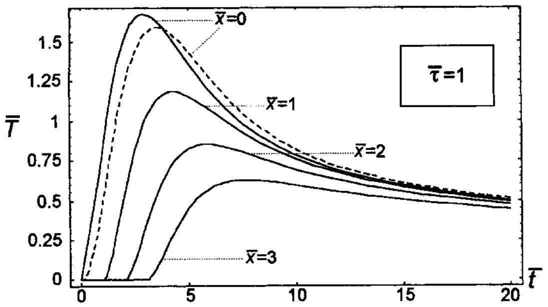

Fig. 6. Plots of $\bar{T}$ vs $\bar{t}$ at different positions and for $\bar{\tau} = 1$, obtained by employing equations (42) and (43); the dashed line corresponds to $\bar{\tau} = 0$ (parabolic case) and $\bar{x} = 0$.

with the local equilibrium hypothesis. In Fig. 6, $\bar{T}$ vs $\bar{t}$ is reported for $\bar{\tau} = 1$ at positions $\bar{x} = 0$, $\bar{x} = 1$, $\bar{x} = 2$ and $\bar{x} = 3$. For the same value of $\bar{\tau}$ and at the same positions, the dimensionless entropy production rate is reported in Fig. 7. The value $\bar{\tau} = 1$ is expected to be the highest value of $\bar{\tau}$ which yields a positive definite entropy production rate. Indeed, Fig. 7 shows that, in this case, the entropy production rate does not present negative values, so that it is conceivable to assume the validity of the local equilibrium scheme. Moreover, Fig. 6 shows that the time-variation of the dimensionless temperature at $\bar{x} = 0$ for $\bar{\tau} = 1$ is different from that for $\bar{\tau} = 0$, i.e. for the case of parabolic heat conduction. In particular, an analysis of the results reported in Fig. 6 shows that the temperature difference at $\bar{x} = 0$ between the cases $\bar{\tau} = 1$ and $\bar{\tau} = 0$ can be greater than $0.37\ Q(\alpha t_p)^{1/2}/k$. For instance, for a material with $k = 1\ \mathrm{W m^{-1} K^{-1}}$ and $\alpha = 10^{-5}\ \mathrm{m^2 s^{-1}}$, a heat flux pulse with $Q = 1\ \mathrm{W\ cm^{-2}}$ and $t_p = 10^{-1}$ s could yield a temperature difference greater than 3.7 K. Moreover, for this material, the propagation speed of thermal waves would be $1\ \mathrm{cm\ s^{-1}}$ in the case $\bar{\tau} = 1$, while it would be infinite in the case $\bar{\tau} = 0$. These circumstances point out the existence of a domain where hyperbolic heat conduction is compatible with the local equilibrium hypothesis and predicts temperature distributions which are definitely different from those predicted by parabolic heat conduction.

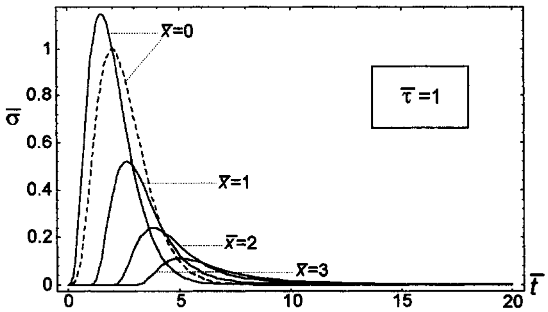

Fig. 7. Plots of $\bar{\sigma}$ vs $\bar{t}$ at different positions and for $\bar{\tau} = 1$, obtained by employing equations (39), (40) and (43); the dashed line corresponds to $\bar{\tau} = 0$ (parabolic case) and $\bar{x} = 0$.

CONCLUSIONS

The compatibility of hyperbolic heat conduction with the hypothesis of local equilibrium has been investigated. It has been pointed out that Cattaneo–Vernotte's equation is compatible with the local equilibrium scheme if the entropy production rate per unit volume is non-negative at every point and at every instant of time. Two problems of thermal wave propagation have been examined: a solid slab which undergoes a sudden temperature change at its boundary; a semi-infinite solid with a time-varying boundary heat flux. In the first problem, the temperature rise in the interior of the slab can be greater than that prescribed at the boundary. The evaluation of the entropy production rate per unit volume, σ, has revealed that the excessive temperature rise within the slab is accompanied by negative values of σ. Therefore, no violation of the second law occurs, because the local equilibrium scheme does not hold and the temperature field cannot be interpreted in the usual thermodynamic sense. The analysis of the second problem has shown that, if the prescribed heat flux at the boundary of the semi-infinite solid behaves as a pulse which decays more rapidly than $e^{-t/\tau}$, then σ presents negative values. In this case, the local equilibrium scheme is not applicable. On the other hand, if the boundary heat flux decay is not steeper than $e^{-t/\tau}$, σ is non-negative at every point and at every instant of time. In this case, the local equilibrium scheme can be applied even if the predicted temperature field can be considerably different from that predicted by Fourier's heat conduction.

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APPENDIX A

Let us find solutions of equations (26) and (28) in the form

$$\vartheta(\xi,\eta) = a(\eta)\cos(\beta\xi). \tag{A1}$$

By substituting equation (A1) in equation (28), one obtains the eigenvalues

$$\beta_n = (2n+1)\pi, \quad n = 0,1,2,\ldots \tag{A2}$$

while, as a consequence of equations (26) and (A2), function $a(\eta)$ must fulfil the differential equation

$$\Lambda\frac{\mathrm{d}^2a}{\mathrm{d}\eta^2}+\frac{\mathrm{d}a}{\mathrm{d}\eta}+\beta_n^2a = 0. \tag{A3}$$

For every value of n, the general solution of equation (A3) can be expressed as

$$a_n(\eta) = b_n\mathrm{e}^{\gamma_n\eta}+c_n\mathrm{e}^{\delta_n\eta} \tag{A4}$$

where the coefficients γ_n and δ_n are given by

$$\gamma_n = -\frac{1}{2\Lambda}(1-\sqrt{1-4\beta_n^2\Lambda}) \tag{A5}$$

$$\delta_n = -\frac{1}{2\Lambda}(1+\sqrt{1-4\beta_n^2\Lambda}). \tag{A6}$$

Therefore, the dimensionless temperature distribution $\vartheta(\xi,\eta)$ can be expressed as

$$\vartheta(\xi,\eta) = \sum_{n=0}^{\infty}(b_n\mathrm{e}^{\gamma_n\eta}+c_n\mathrm{e}^{\delta_n\eta})\cos(\beta_n\xi). \tag{A7}$$

By substituting equation (A7) into equation (27), one obtains

$$\sum_{n=0}^{\infty}(b_n+c_n)\cos(\beta_n\xi) = 1 \tag{A8}$$

$$\sum_{n=0}^{\infty}(b_n\gamma_n+c_n\delta_n)\cos(\beta_n\xi) = 0. \tag{A9}$$

On account of equation (A2) and of the orthogonality relation

$$\int_{-1/2}^{1/2}\cos((2m+1)\pi\xi)\cos((2n+1)\pi\xi)\,\mathrm{d}\xi = \begin{cases}1/2, & m=n\\ 0, & m\neq n\end{cases} \tag{A10}$$

equations (A8) and (A9) can be employed to obtain the constants b_n and c_n, namely

$$b_n = (-1)^n\frac{4\delta_n}{\beta_n(\delta_n-\gamma_n)} \tag{A11}$$

$$c_n = (-1)^{n+1}\frac{4\gamma_n}{\beta_n(\delta_n-\gamma_n)}. \tag{A12}$$

It can be easily checked that the dimensionless temperature field is a continuous function of the parameter Λ, even for the values of Λ such that the right hand sides of equations (A11) and (A12) become singular.

APPENDIX B

Equations (35)–(37) can be solved by the Laplace transform method. The Laplace transform of $q(x,t)$ is given by

$$\tilde{q}(x,p) = \int_0^{\infty}\mathrm{e}^{-pt}q(x,t)\,\mathrm{d}t. \tag{B1}$$

As a consequence of equation (36) and of the properties of Laplace transforms (see for instance Churchill ref. [20]), equations (35) and (37) can be rewritten as

$$\alpha\frac{\mathrm{d}^2\tilde{q}}{\mathrm{d}x^2} = (p+\tau p^2)\tilde{q} \tag{B2}$$

$$\tilde{q}(0,p) = Q\tilde{F}(p). \tag{B3}$$

The solution of equations (B2) and (B3) which is bounded for $x \to +\infty$ can be expressed as

$$\tilde{q}(x,p) = Q\tilde{F}(p)\tilde{g}(x,p) \tag{B4}$$

where $\tilde{g}(x,p)$ is given by

$$\tilde{g}(x,p) = \mathrm{e}^{-x\sqrt{p(1+\tau p)}/\sqrt{\alpha}}. \tag{B5}$$

On account of the convolution property of inverse Laplace transforms [20], equation (B4) yields

$$q(x,t) = Q\int_0^t F(t-t')g(x,t')\,\mathrm{d}t'. \tag{B6}$$

Equation (B5) can be rewritten as

$$\tilde{g}(x,p) = -\sqrt{\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{e}^{-x\sqrt{p(1+\tau p)}/\sqrt{\alpha}}}{\sqrt{p(1+\tau p)}} \tag{B7}$$

so that

$$g(x,t) = -\sqrt{\alpha}\frac{\mathrm{d}}{\mathrm{d}x}\mathscr{L}^{-1}\left\{\frac{\mathrm{e}^{-x\sqrt{p(1+\tau p)}/\sqrt{\alpha}}}{\sqrt{p(1+\tau p)}}\right\}. \tag{B8}$$

The inverse Laplace transform which appears in equation (B8) is given by [21]

$$\mathscr{L}^{-1}\left\{\frac{\mathrm{e}^{-x\sqrt{p(1+\tau p)}/\sqrt{\alpha}}}{\sqrt{p(1+\tau p)}}\right\} = \frac{e^{-t/(2\tau)}}{\sqrt{\tau}}U\left(t-x\sqrt{\frac{\tau}{\alpha}}\right)I_0\left(\frac{1}{2\tau}\sqrt{t^2-\frac{x^2\tau}{\alpha}}\right). \tag{B9}$$

By employing equations (B8)–(B9) and the properties of modified Bessel functions [22], one obtains

$$g(x,t) = \sqrt{\frac{\alpha}{\tau}}\mathrm{e}^{-t/(2\tau)}\left[\frac{xU\left(t-x\sqrt{\frac{\tau}{\alpha}}\right)}{2\alpha\sqrt{t^2-\frac{x^2\tau}{\alpha}}}I_1\left(\frac{1}{2\tau}\sqrt{t^2-\frac{x^2\tau}{\alpha}}\right)\right.$$

$$\left.+\sqrt{\frac{\tau}{\alpha}}\delta\left(t-x\sqrt{\frac{\tau}{\alpha}}\right)I_0\left(\frac{1}{2\tau}\sqrt{t^2-\frac{x^2\tau}{\alpha}}\right)\right] \tag{B10}$$

By substituting equation (B10) in equation (B6), the heat flux density can be expressed as

$$q(x,t) = QU\left(t-x\sqrt{\frac{\tau}{\alpha}}\right)$$

$$\times\left[\frac{x}{2\sqrt{\alpha\tau}}\int_{x\sqrt{\tau/\alpha}}^{t}\frac{\mathrm{e}^{-t'/(2\tau)}F(t-t')}{\sqrt{t'^2-\frac{x^2\tau}{\alpha}}}I_1\left(\frac{1}{2\tau}\sqrt{t'^2-\frac{x^2\tau}{\alpha}}\right)\mathrm{d}t'\right.$$

$$\left.+F\left(t-x\sqrt{\frac{\tau}{\alpha}}\right)\mathrm{e}^{-x/(2\sqrt{\alpha\tau})}\right] \tag{B11}$$